2017 TIANYUAN SPRING SCHOOL

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1. Exercises for *Minimal model program and its applications in higher dimensional geometry*

Question 1.1. *Let S be the blow up of a point $p \in \mathbb{P}^2$.*

(1) *Compute the generators of $NE(S)$.*
(2) *Compute the shape of $Nef(S)$ under the identification of $N_1(S) = N^1(S)$.*

Question 1.2. *Let X be the blow up of a point $p \in \mathbb{P}^3$.*

(1) *Compute the generators of $NE(X)$.*
(2) *Compute the shape of $Nef(X)$ under suitable natural basis.*

Question 1.3. *Let X be a surface such that $K_X \cong \mathcal{O}_X$ e.g., X is K3-surface or Abelian surface. L is a big and nef line bundle on X. Prove $h^0(X, L) \geq 2$.*

Question 1.4. *Let X be a projective variety with Gorenstein singularities. Assume X has a crepant resolution, i.e. there is a resolution $f : Y \to X$ such that $f^*(\omega_X) = \omega_Y$. Then prove that for a big and nef line bundle L on X, $H^i(X, \omega_X \otimes L) = 0$.*

Question 1.5 (Examples of round cones)**.** *Exercise 32 in* [Kol08].

Question 1.6. *Prove the surface singularities $xy = z^{n+1}$ of type A_n is canonical.*

Question 1.7. *Let (X, Δ) be a pair such that X is normal variety and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ is a finite dominant morphism from a normal variety. Define Δ_Y by*

$$f^*(K_X + \Delta) = K_Y + \Delta_Y.$$

Then $a(E, X, \Delta) \geq -1$ (resp. > -1) for any divisor E if and only if $a(E_Y, Y, \Delta_Y) \geq -1$ (resp. > -1) for any divisor E_Y.

Question 1.8. *Let (X, Δ) be a klt pair, then show the set*

$$\{E \mid a(E, X, \Delta) < 0\}$$

is a finite set.

Question 1.9 (Cone singularities)**.** *Exercise 70 in* [Kol08].

Question 1.10. *Let (X, Δ) be a n-dimensional pair such that X is normal variety and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Fix a point x on X, we define $\mathrm{mld}_x(X, \Delta)$ to be*

$$\min\{a(X, \Delta, E) + 1 | \mathrm{Center}_X(E) = x\}.$$

1. Prove if $x \in X$ is smooth, then $\mathrm{mld}_x(X) = n$.
2^*. *Prove that for any $\Delta \geq 0$ and $x \in X$, $\mathrm{mld}_x(X, \Delta) \leq n$ and the equality holds if and only if X is smooth at x and $\mathrm{Supp}(\Delta)$ does not contain x.*

Remark: Part 2 is open except in lower dimension. In fact, for a fixed dimension n, it's not even know that $\mathrm{mld}_x(X,\Delta)$ is bounded from above by a number which only depends on n.

Question 1.11. *Let $f\colon X\to Y$ be a birational morphism from a projective smooth surface to a normal surfae. Let $p\in X$ and $f^{-1}(p)=\sum_{i=1}^m E_i$, then prove the matrix $(g_{ij})_{1\le i\le m}$ whose entries $g_{ij}=E_i\cdot E_j$ is negative definite.*

Question 1.12. *(1) Let $f\colon X\to Y$ be a projective birational morphism from a smooth rational surface to a normal surface. Assume $f(E)$ is a point for a curve E on Y, assume $K_X\cdot E<0$, then prove E is a (-1)-curve.*
(2) Use this to prove that given a normal surface singularity $y\in Y$, among all its resolutions, there is a unique minimal resolution $f\colon X\to Y$ such that $K_X\cdot E\ge 0$, for any E with $f(E)=y$.

Question 1.13. *Prove Theorem 3 in* [Kol08].

Question 1.14. *Exercise 19 in* [Kol08].

Question 1.15. *Exercise 38-46 in* [Kol08].

The following facts have been used a few times in our argument. It has a central importance in higher dimensional geometry.

Question 1.16. *Let (X,Δ) be a lct pair and $M\neq 0$ be a Cartier divisor.*

(1) *if M is a general member in a base point free linear system $|L|$, then for any $c<1$, $(X,\Delta+cM)$ is klt.*
(2) *Define the* **log canonical threshold** *of (X,Δ) with respect to M to be*
$$\mathrm{lct}(X,\Delta;M)=\max\{t|K_X+\Delta+tM \textit{ is log canonical}\}.$$
Show $\mathrm{lct}(X,\Delta;M)\in(0,\infty)$.
(3) *Assume a linear system $|L|$ has base locus B. Let $M_1,...,.M_m$ be m general divisors in $|L|$. Let $c=\mathrm{lct}(X,\Delta;\sum_{i=1}^m M_i)$. Show that for $m\gg 0$, the divisor E such that*
$$a(E,X,\Delta+c\sum_{i=1}^m M_i)=-1$$
has its center contained in B. (Such center is called a **log canonical center***).*

Question 1.17 (Tie-and-break)**.** *Let (X,Δ) be a klt pair, M a big divisor. Let $c=\mathrm{lct}(X,\Delta;M)$. Prove that for any $\epsilon>0$, we can always find a $\mathbb{Q}$-divisor M_1 such that $M_1\sim_{\mathbb{Q}} c_1M$ with $|c-c_1|<\epsilon$ and $(X,\Delta+c_1M_1)$ is log canonical and has precisely one divisor E such that $a(E,X,\Delta+c_1M_1)=-1$.*

Question 1.18. *Show that if X and Y are birational, and both of them have canonical singularities, then*
$$H^0(X,mK_X)\cong H^0(Y,mK_Y).$$

Question 1.19. *Let $X\to C$ be a projective semistable family over a smooth curve compactifying a smooth family $X^0\to C^0=C\setminus\{p\}$. Let $\pi\colon B\to C$ be a finite morphism between smooth curves, and let X_B be a projective semistable family which compactifies the family $X^0\times_{C^0}B^0$ where $B^0=\pi^{-1}(C^0)$. Prove*
$$\pi^*R(X/C,K_{X/C})=R(X_B/B,K_{X_B/B}).$$

Question 1.20. *Use conclusions in the above questions to show that the KSBA limit doesn't depend on the semistable reduction.*

Question 1.21 (Kollár-Shokurov Connectedness Theorem)**.** *Let (X, D) be a log pair, i.e., X is normal, D is an effective $\mathbb{Q}$-divisor. Assume $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to (X, D)$ be a log resolution and write*

$$f^*(K_X + D) + \sum_{i, a_i > -1} a_i E_i + \sum_{j, b_j \leq -1} b_j F_j = K_Y,$$

where E_i and F_j does not have common components. Then prove that $\mathrm{Supp} F = \sum_j F_j$ *is connected in a neighborhood of any fiber of f.*

References

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.

[Kol08] János Kollár, *Exercises in the birational geometry of algebraic varieties*, arXiv:0809.2549 (2008).

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