EXERCISES ON MULTIPLIER IDEALS

Exercice 1. Assume that X is a smooth variety and D an effective $\mathbb{Q}$-divisor. Let $\Sigma_k(D) = \{x \in D \mid \mathrm{mult}_x D \geq k\}$. Show that if (X, D) is log-canonical, then $\mathrm{codim}_X \Sigma_k(D) \geq k$ and if .(X, D) is KLT, then $\mathrm{codim}_X \Sigma_k(D) > k$.

Exercice 2. Let X be a smooth projective surface and let B be a big and nef $\mathbb{Q}$-divisor on X. Then $H^i(X, \mathcal{O}_X(K_X + \lceil B \rceil)) = 0$ for $i > 0$. (Sakai's lemma)

Exercice 3. Let (A, Θ) be a principally polarized abelian variety. For some integer $m \geq 1$, fix $D \in |m\Theta|$. Prove that $(A, \frac{1}{m}D)$ is log-canonical.

Exercice 4. Let D be a reduced integral divisor on a smooth variety X. Fix a log resolution $\mu : X' \to X$ of (X, D) such that the proper transform D' of D is non-singular and write $\mu^* D = D' + F$, where F is a μ-exceptional divisor on X'. The induced morphism $\nu : D' \to D$ is a resolution of singularities of D. Define the adjoint ideal $\mathrm{adj}(D) \subset \mathcal{O}_X$ to be the ideal sheaf

$$\mathrm{adj}(D) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - F).$$

1) $\mathrm{adj}(D)$ is independent of the choice of log resolution.
2) Show that we have an exact sequence

$$0 \to \mathcal{O}_X(K_X) \xrightarrow{\cdot D} \mathcal{O}_X(K_X + D) \otimes \mathrm{adj}(D) \to \nu_* \mathcal{O}_{D'}(K_{D'}) \to 0.$$

3) Show that $\mathrm{adj}(D) = \mathcal{O}_X$ if and only if D is normal and has canonical singularities (or rational singularities).

Exercice 5. Let (A, Θ) be a principally polarized abelian variety .
1) Assume that Θ is irreducible. Admit the fact that for any desingularization $\nu : Y \to \Theta$ and $P \in \mathrm{Pic}^0(A)$, $h^0(Y, (K_Y) \otimes P) > 0$, show that Θ has at worst rational singularities. (Ein-Lazarsfeld)
2) It is known that a general (A, Θ) can be decomposed uniquely as the product of irreducible principally polarized abelian varieties $(A, \Theta) \simeq (A_1, \Theta_1) \times \cdots \times (A_r, \Theta_r)$. Show that if for some $k \geq 2$, $\mathrm{codim}_A \Sigma_k(\Theta) = k$, then (A, Θ) splits as a k-fold product of PPAVs. (Ein-Lazarsfled, [EL])
2) Let $D \in |m\Theta|$ such that $\lfloor \frac{1}{m}D \rfloor = 0$. Show that $(X, \frac{1}{m}D)$ is log-terminal. (Hacon, difficult, need knowledge of generic vanishing, [Hacon])

Références

[EL] Ein, L., Lazarsfeld, R., Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. **10** (1997), no. 1, 243–258.

[Hacon] Hacon, Ch., Divisors on principally polarized abelian varieties, Compositio Mathematica **119**, no.3, 321–329.